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RESEARCH REPORT No. EM-53

DIFFRACTION OF PULSES BY PARABOLIC CYLINDERS AND PARABOLOIDS OF REVOLUTION

by

IRVIN KAY

CONTRACT No. AF-19(122)-42

New York University
Washington Square College of Arts and Science
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The research reported in this document has been made possible through support and sponsorship extended by the Geophysics Research Directorate of the Air Force Cambridge Research Center, under Contract No. AF-19(122)-42. It is published for technical information only, and does not necessarily represent recommendations or conclusions of the sponsoring agency.

June, 1953

ABSTRACT

The diffraction of plane pulses by a parabolic cylinder and by a paraboloid of revolution is investigated for both the boundary conditions $u = 0$ and $\frac{\partial u}{\partial n} = 0$. In all cases the pulse is assumed to be incident on the vertex of the diffracting obstacle in a direction parallel to its axis. The problems are treated by first separating variables in the time dependent wave equation and then solving the resulting integral equations by means of a Laplace transform. In each case a Volterra integral equation for the solution is obtained, which can be solved by the usual iteration process. It is shown that when the parabolic cylinder degenerates into a half plane, the solution reduces to that obtained for the corresponding problem by other methods.

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Introduction

For mathematical as well as physical reasons it seems convenient to divide diffraction problems that arise in the theories of wave propagation governed by the wave equation

$$\Delta u - \frac{1}{c^2} u_{tt} = 0$$

into two classes: time harmonic problems in which incident time harmonic wave trains are considered, and problems in which the dependence upon time of the functions involved is not periodic. In the latter class a typical case is that of the incident pulse or transient wave. A pulse wave may be defined for the purposes of this article as a moving discontinuity surface behind which there is a prescribed field varying in space and time in such a way as to satisfy the wave equation, and in front of which the field is zero. A diffraction problem arises when an obstacle is assumed to exist in the path of the moving pulse wave and the field for all time is constrained to satisfy certain homogeneous boundary conditions on the surface of the obstacle. In addition, the physically reasonable condition is usually imposed, at least by implication, that the total field be a limit of continuous solutions of the wave equation and that a first time derivative of the field be a limit of the continuous first time derivatives of continuously differentiable solutions of the wave equation.* The diffraction problem then is to determine the motion of the discontinuity surface and the field generated for all time and at all points of space where the field is continuous.

In report No. EM-43[7] the problem of determining the field due to an electromagnetic pulse wave of arbitrary shape incident on a perfectly conducting infinite wedge such that the electric vector or the magnetic vector is polarized parallel to the generators of the wedge was treated. In this report we shall consider the diffraction of a plane pulse wave incident on the vertex of a perfectly conducting parabolic cylinder in a direction parallel to the axis of the parabolic cross section of the cylinder. We shall again consider two cases: one in which the electric vector of the incident field is assumed to be polarized in a direction parallel to the generators of the parabolic cylinder, and another in which the magnetic vector of the incident field is assumed to have that polarization. In these cases the problem of determining the field reduces to solving a boundary value problem for the scalar wave equation

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* For a discussion of the implications of such a requirement for a solution of a hyperbolic partial differential equation, the conditions which the discontinuity of the solution must satisfy as well as the manner in which it must propagate, see [1], p. 356 ff. and p. 470, and [4]. For a similar discussion of a discontinuous solution of Maxwell's equations in electromagnetic theory, see [2] and [3].

$$u_{xx} + u_{yy} - \frac{1}{c^2} u_{tt} = 0$$

where, for case 1, $u = 0$ on the parabolic cylinder for all time, and for case 2, $\frac{\partial u}{\partial n} = 0$ on the parabolic cylinder for all time; $\frac{\partial}{\partial n}$ refers to a derivative taken in a direction normal to the surface of the parabolic cylinder.

We shall also consider, for the sake of completeness, the analogous boundary value problems for a plane pulse similarly incident on a paraboloid of revolution.

Both of these problems in the case of sound wave propagation have been treated by Friedlander [5] and by Chester [6]. Chester obtained solutions in the form of integrals which are not generally suitable for numerical computation. Friedlander reduced the problem to one for a Volterra integral equation, the solution of which can be obtained by an iteration procedure. In this report we have obtained results essentially equivalent to those of the above authors by a different method which is analogous to the method used in Report No. EM-43 for the infinite wedge.

1. The Parabolic Cylinder

1.1 Formulation of the Characteristic Boundary Value Problem

We shall be concerned in this section with the field resulting from the diffraction of a uniform plane pulse wave incident on the convex side of a parabolic cylinder and traveling at the propagation speed 1 in a direction parallel to the axis of the parabolic cross section. We shall have to deal with the wave equation

$$u_{xx} + u_{yy} - u_{tt} = 0. \tag{1.11}$$

The incident field is taken to be 1 on the initial side of the plane discontinuity surface which characterizes the incident pulse wave, and it is taken to be zero on the other side.

The equation of the parabolic cylinder in Cartesian coordinates is taken to be

$$x^2 = 4p(p-y) \tag{1.12}$$

where p is the focal length of the parabolic cross section.

In (x,y,t) space the incident wave front is a plane slanted at 45 degrees from the horizontal (x,y) plane. Through the curve of intersection of the incident wave front and the parabolic cylinder in (x,y,t) space there passes another characteristic surface of (1.11). This second characteristic surface, representing the space-time history of the reflected wave front, is a characteristic cone whose vertex lies on the focal line of the cylinder. The fact that the cone passes through the curve of intersection of the incident plane and the parabolic cylinder can be seen easily from the relations $t=p-y$ for the incident wave, $x^2=4p(p-y)$ for the parabolic cylinder, and $t+p=\sqrt{x^2+y^2}$ for the characteristic cone; we assume that the incident wave first strikes the cylinder at time $t=0$.

In place of Cartesian coordinates we shall use a system such that the characteristic cone representing the reflected wave front in (x,y,t) space is a constant coordinate surface and the parabolic cylinder representing the diffracting obstacle is another. If $t=0$ is the time at which the incident wave strikes the vertex of the parabola, we have then

$$t = \frac{1}{2}(\xi^2+\eta^2) - (\phi+p) \qquad \phi = \sqrt{x^2+y^2}-(t+p)$$

$$y = \frac{1}{2}(\eta^2-\xi^2) \qquad \xi^2 = \sqrt{x^2+y^2}-y$$

$$x = \xi\eta \qquad \eta^2 = \sqrt{x^2+y^2}+y. \tag{1.13}$$

ξ and η are the usual parabolic coordinates. η takes only positive values, but ξ may be either positive or negative. In these coordinates the cylinder (1.12) is given by

$$\eta = \sqrt{2p}. \tag{1.14}$$

Employing the transformation (1.13), equation (1.11) becomes

$$u_{\xi\xi} + u_{\eta\eta} + 2\xi u_{\phi\xi} + 2\eta u_{\phi\eta} + 2u_{\phi} = 0. \tag{1.15}$$

Now if the reflected wave front is represented by the expression $\Psi(x,y)=t$, then $\Psi(x,y)$ must satisfy the eiconal equation

$$(\nabla\Psi)^2=1.$$

Since the characteristic cone whose spatial cross sections are the various positions of the reflected wave front can be represented by $t+p=\sqrt{x^2+y^2}$, this requirement has been met. The discontinuity on the reflected wave front must have the form[1],[2],[3],[4]

$$[u] = U_0 e^{-\frac{1}{2}\int_0^s(\Delta\Psi)ds},$$

where U_0 is some initial value on each ray. If we use polar coordinates (r,θ) instead of Cartesian coordinates, we can write for the discontinuity

$$[u] = U_0 e^{-\frac{1}{2}\int^r \Delta\, r dr} = \frac{f(\theta)}{\sqrt{r}}, \tag{1.16}$$

where $f(\theta)$ is a function to be determined by the boundary condition on the parabolic cylinder.

In polar coordinates the equation (1.14) for the parabolic cylinder becomes

$$r = \frac{2p(1-\sin\theta)}{\cos^2\theta} = \frac{2p}{1+\sin\theta}.$$

The first boundary value problem is that in which $u = 0$ on the diffracting obstacle, or $u(\xi, \sqrt{2p}, \emptyset) = 0$. Then we have $\frac{f(\theta)}{\sqrt{r(\theta)}} = f(\theta)\sqrt{\frac{1+\sin\theta}{2p}} = 1.$ Thus $f(\theta) = \sqrt{\frac{2p}{1+\sin\theta}}$. It follows from (1.16) and the fact that the incident field is one behind the incident wave front, that the inhomogeneous boundary condition on the inside of the reflected wave front is

$$u(\xi,\eta,0) = 1 - \sqrt{\frac{2p}{r(1+\sin\theta)}} ;$$

$\emptyset = 0$ is the equation of the characteristic cone which represents the reflected wave front. In parabolic coordinates this condition is

$$u(\xi,\eta,0) = 1 - \frac{\sqrt{2p}}{\eta} . \tag{1.17}$$

The first characteristic boundary value problem can now be stated: we must find a solution $u(\xi,\eta,\emptyset)$ of (1.15) such that $u(\xi,\eta,0) = 1 - \frac{\sqrt{2p}}{\eta}$ and $u(\xi,\sqrt{2p},\emptyset) = 0$ in the region determined by $-\infty < \xi < \infty$, $\sqrt{2p} \leq \eta < \infty$, and $-\infty < \emptyset \leq 0$.

The boundary condition $\frac{\partial u}{\partial n} = 0$ implies that the field inside the wave front at the parabolic cylinder is twice the incident field at that point, or 2. Thus, in this case we have

$$u(\xi,\eta,0) = 1 + \frac{\sqrt{2p}}{\eta} . \tag{1.18}$$

In the coordinate system given by (1.13) we have on the parabolic cylinder

$$\left.\frac{\partial u}{\partial n}\right|_{\eta=\sqrt{2p}} = \sqrt{2p}\left.\frac{\partial u}{\partial \emptyset}\right|_{\eta=\sqrt{2p}} + \left.\frac{\partial u}{\partial \eta}\right|_{\eta=\sqrt{2p}} = 0 . \tag{1.19}$$

Thus the second characteristic boundary value problem is the following: we must find a solution $u(\xi,\eta,\emptyset)$ of (1.15) such that $u(\xi,\eta,0) = 1 + \frac{\sqrt{2p}}{\eta}$ and

$$\sqrt{2p}\left.\frac{\partial u}{\partial \emptyset}\right|_{\eta=\sqrt{2p}} + \left.\frac{\partial u}{\partial \eta}\right|_{\eta=\sqrt{2p}} = 0$$

in the region determined by $-\infty < \xi < \infty$, $\sqrt{2p} \leq \eta < \infty$, and $-\infty < \emptyset \leq 0$.

1.2 Solution of the First Characteristic Boundary Value Problem

The boundary condition (1.17) involves the variable η alone. The boundary condition on the parabolic cylinder is zero. Thus, it seems reasonable to seek a solution which is independent of ξ. If we assume that our solution is independent of ξ, the equation (1.15) takes the form

$$u_{\eta\eta} + 2\eta u_{\phi\eta} + 2u_{\phi} = 0. \tag{1.21}$$

We must find $u(\eta,\phi)$ satisfying (1.21) in the region $-\infty < \phi \leq 0$,$\sqrt{2p} \leq \eta < \infty$ with $u(\sqrt{2p},\phi) = 0$ and $u(\eta,0) = 1 - \frac{\sqrt{2p}}{\eta}$.

(1.21) separates into

$$\frac{d\Phi(\phi)}{d\phi} + k\Phi(\phi) = 0$$

and

$$\frac{d^2H_k(\eta)}{d\eta^2} - 2k\eta\frac{dH_k(\eta)}{d\eta} - 2kH_k(\eta) = 0, \tag{1.22}$$

where k is the separation constant, Φ is a function of ϕ alone, and H_k is a function of η alone. We assume a solution which is a linear combination of solutions of the equations (1.22) in product pairs:

$$u(\eta,\phi) = \int_L A(k)\, H_k(\eta)e^{-k\phi}\, dk , \tag{1.23}$$

where A(k) and the contour L are to be determined.

Since $u(\sqrt{2p},\phi) = 0$, it is required that $H_k(\sqrt{2p}) = 0$. Moreover, from

$u(\eta,0) = 1 - \frac{\sqrt{2p}}{\eta}$ we have

$$\int_L A(k)\, H_k(\eta)dk = 1 - \frac{\sqrt{2p}}{\eta}. \tag{1.24}$$

The solution of the second equation in (1.22) which satisfies $H_k(\sqrt{2p}) = 0$ is

$$H_k(\eta) = e^{k\eta^2}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2}\, d\tau . \tag{1.25}$$

We now have, combining (1.25) and (1.23) and introducing a factor 2 for future convenience,

$$u(\eta,\phi) = 2\int_L A(k)e^{k(\eta^2-\phi)}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2}\, d\tau\, dk. \tag{1.26}$$

We require, according to (1.24),

$$u(\eta,0) = 2\int_L A(k)e^{k\eta^2}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2}\,d\tau dk = 1 - \frac{\sqrt{2p}}{\eta}$$

or, setting $\eta^2 = x$,

$$\int_L A(k)e^{kx}\int_{2p}^{x} \frac{e^{-k\tau}}{\sqrt{\tau}}\,d\tau\,dk = 1 - \sqrt{\frac{2p}{x}}\,.$$

Then, changing variables, we have

$$\int_L A(k)e^{k(x-2p)}\int_0^{x-2p} \frac{e^{-k\tau}}{\sqrt{\tau+2p}}\,d\tau\,dk = 1 - \sqrt{\frac{2p}{x}}\ ,$$

and, setting $x-2p = z$, we find

$$\int_L A(k)e^{kz}\int_0^{z} \frac{e^{-k\tau}}{\sqrt{\tau+2p}}\,d\tau\,dk = 1 - \sqrt{\frac{2p}{z+2p}}\ .$$

This suggests that we might make L the contour $(c - i\infty, c + i\infty)$ used in the inversion theorem for the Laplace transform. Continuing formally, we have

$$\int_0^{z} \frac{d\tau}{\sqrt{\ +2p}}\int_L A(k)e^{k(z-\tau)}\,dk = 1 - \sqrt{\frac{2p}{z+2p}}\ ,$$

and we observe that the left side is a convolution integral for the Laplace transform. Thus, taking Laplace transforms of both sides, we have

$$A(k)\int_0^{\infty} \frac{e^{-k\tau}}{\sqrt{\tau+2p}}\,d\tau = \frac{1}{k} - \sqrt{2p}\int_0^{\infty}\frac{e^{-k\tau}}{\sqrt{\tau+2p}}\,d\tau. \qquad (1.27)$$

Finally,

$$A(k) = \frac{1}{k\int_0^{\infty}\frac{e^{-k\tau}}{\sqrt{\tau+2p}}\,d\tau} - \sqrt{2p}\ .$$

We introduce the notation $L_k\left\{f(\tau)\right\} = \int_0^\infty f(\tau)e^{-k\tau}\,d\tau$ for the Laplace transform. Then, substituting (1.27) in (1.26) with the appropriate change of variables, we obtain

$$u(\eta,\phi) = \int_L A(k)e^{k(x-\phi)}\int_{2p}^{x}\frac{e^{-k\tau}}{\sqrt{\tau}}\,d\tau dk = \int_L\left\{\frac{1}{kL_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}}-\sqrt{2p}\right\}e^{k(x-\phi)}\int_{2p}^{x}\frac{e^{-k\tau}}{\sqrt{\tau}}\,d\tau dk$$

$$=\int_L\left\{\frac{1}{kL_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}}-\sqrt{2p}\right\}e^{k(x-\phi)}\left\{e^{-2pk}L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}-e^{-kx}L_k\left\{\frac{1}{\sqrt{\tau+x}}\right\}\right\}dk$$

$$=\int_L\frac{e^{k(x-\phi-2p)}}{k}\,dk-\sqrt{2p}\int_L e^{k(x-\phi-2p)}\,L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}dk-\int_L\frac{e^{-k\phi}L_k\left\{\frac{1}{\sqrt{\tau+x}}\right\}}{kL_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}}\,dk$$

$$+\sqrt{2p}\int_L e^{-k\phi}\,L_k\left\{\frac{1}{\sqrt{\tau+x}}\right\}dk\,.$$

The first, second, and fourth integrals are simply inversions of the Laplace transforms which appear in their integrands. The second and fourth integrals cancel, and the first integral has the value 1. Thus

$$u(\eta,\phi) = 1 - I(-\phi), \tag{1.28}$$

where

$$I(y) = \int_L\frac{e^{ky}L_k\left\{\frac{1}{\sqrt{\tau+x}}\right\}}{kL_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}}\,dk. \tag{1.29}$$

We now take the Laplace transform of both sides of (1.29) and obtain

$$L_k\left\{I(\tau)\right\} = \frac{1}{k}\,\frac{L_k\left\{\frac{1}{\sqrt{\tau+x}}\right\}}{L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}},$$

or

$$L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}L_k\left\{I(\tau)\right\} = L_k\{1\}\;L_k\left\{\frac{1}{\sqrt{\tau+x}}\right\}.$$

Thus, by the faltung theorem,

$$\int_0^y \frac{I(\tau)d\tau}{\sqrt{y+2p-\tau}} = \int_0^y \frac{d\tau}{\sqrt{\tau+x}} .$$

Differentiating both sides with respect to y we have

$$I(y) - \sqrt{\frac{p}{2}} \int_0^y \frac{I(\tau)d\tau}{(y+2p-\tau)} = \sqrt{\frac{2p}{x+y}} . \tag{1.210}$$

Instead of attempting to justify the formal derivation of (1.210), one can show directly that $u(\eta,\phi)$, given by (1.28), with $I(-\phi)$ obtained by solving the integral equation (1.210), satisfies all the conditions of the problem of which $u(\eta,\phi)$ is a solution.

If we set $y=0$, we obtain $I(0) = \sqrt{\frac{2p}{x}} = \frac{\sqrt{2p}}{\eta}$. This verifies the fact that $u(\eta,0) = 1 - \frac{\sqrt{2p}}{\eta}$. To obtain the other boundary condition $u(\sqrt{2p},\phi) = 0$ we integrate (1.210) from 0 to y on both sides of the equation and obtain again

$$\int_0^y \frac{I(\tau)\, d\tau}{\sqrt{y+2p-\tau}} = \int_0^y \frac{d\tau}{\sqrt{\tau+x}}$$

which can be written

$$\int_0^y \frac{I(\tau)d\tau}{\sqrt{y+2p-\tau}} = \int_0^y \frac{d\tau}{\sqrt{y+\eta^2-\tau}} .$$

Putting $\eta = \sqrt{2p}$ we obtain

$$\int_0^y \frac{[I(\tau)-1]}{\sqrt{y+2p-\tau}} d\tau = 0.$$

Differentiating this integral with respect to y we obtain

$$\frac{I(y)-1}{\sqrt{2p}} - \frac{1}{2}\int_0^y \frac{[I(\tau)-1]}{(y+2p-\tau)^{3/2}} d\tau = 0;$$

Hence $I(0) = 1$. Now, integrating by parts, we obtain

$$\int_0^y \frac{I'(\tau)d\tau}{\sqrt{y+2p-\tau}} = 0,$$

which is of the same form as the original integral, except that we have $I'(\tau)$ instead of $I(\tau)-1$. In the same manner we can show that all derivatives of $I(y)$ are zero. Since we shall consider only a regular solution for $I(y)$, it follows that $I(y) = 1$ when $\eta = \sqrt{2p}$; i.e., $u(\sqrt{2p}, \phi) = 0$, and our homogeneous boundary condition is satisfied.

Similarly, since the right side of (1.210) satisfies (1.21) and because of the boundary condition on I at zero, we can show

$$\int_0^y \frac{(I_{\eta\eta} + 2\eta I_{\eta\phi} + 2I_\phi)}{(-\phi + 2p - \tau)^{3/2}} \, d\tau = 0.$$

From the regularity of I it follows that I satisfies(1.21).

We can solve (1.210) by iteration, setting

$$I_1 = \sqrt{\frac{2p}{x+y}} + \sqrt{\frac{p}{2}} \int_0^y \frac{I_0(\tau) d\tau}{(y+2p-\tau)^{3/2}}$$

$$I_2 = \sqrt{\frac{2p}{x+y}} + \sqrt{\frac{p}{2}} \int_0^y \frac{I_1(\tau) d\tau}{(y+2p-\tau)^{3/2}}, \ldots,$$

$$I_n = \sqrt{\frac{2p}{x+y}} + \sqrt{\frac{p}{2}} \int_0^y \frac{I_{n-1}(\tau) d\tau}{(y+2p-\tau)^{3/2}}.$$

We have then $|I_{n+1} - I_n| = \sqrt{\frac{p}{2}} \left| \int_0^y \frac{I_n(\tau) - I_{n-1}(\tau)}{(y+2p-\tau)^{3/2}} d\tau \right| \leq \frac{1}{4p} \int_0^y |I_n - I_{n-1}| d\tau.$

If we assume $y \geq 0$, as it is for all points exterior to the parabolic cylinder, and set $I_0 = 0$, we have

$$|I_{n+1} - I_n| \leq \frac{1}{(4p)^n} \frac{\sqrt{2p}}{\eta} \frac{y^n}{n!}.$$

Thus $I_n = \sum_{i=0}^{n-1} (I_{i+1} - I_i)$ approaches a limit dominated by $\frac{\sqrt{2p}}{\eta} \sum_{n=0}^{\infty} \frac{1}{(4p)^n} \frac{y^n}{n!} = \frac{\sqrt{2p}}{\eta} e^{y/4p}$ which converges rapidly as long as y is not >> p. Moreover, since the I_n are all regular and uniformly bounded in any closed finite region for $y >> 0$, the limit I is a regular function in any such region.

It can be seen that the iterations are not difficult to carry out, since the integrations involved are elementary. For example, we have

$$I_1(y) = \sqrt{\frac{2p}{x+y}}\ , \text{ and } I_2(y) = \frac{\sqrt{2p(x+y)}}{2p+x+y} - \frac{2p}{2p+x+y}\sqrt{\frac{x}{2p+y}} + \sqrt{\frac{2p}{x+y}}\ ,$$

so that for two iterations,

$$u(\eta,\phi) \sim 1 + \frac{2\eta}{(2p+\eta^2-\phi)\sqrt{2p-\phi}} - \sqrt{\frac{2p}{\eta^2-\phi}} - \frac{\sqrt{2p(\eta^2-\phi)}}{2p+\eta^2-\phi}\ . \qquad (1.211)$$

This is a good approximation, at least if $-\phi < 4p$, i.e., if the field is measured at a point not too long after a wave front has passed.

For the case p=0, the parabolic cylinder degenerates into the half plane and (1.29) becomes

$$I = \frac{1}{\sqrt{\pi}} \int_L \frac{e^{ky}}{\sqrt{k}} \int_0^\infty \frac{e^{-k\tau}}{\sqrt{\tau+x}}\, d\tau\, dk = \frac{1}{\pi}\int_0^y \frac{d\tau}{\sqrt{y-\tau}\ \sqrt{\tau+x}} = 1 - \frac{2}{\pi}\tan^{-1}\sqrt{\frac{x}{y}}, \quad (1.212)$$

where we again make use of the *faltung* theorem.

This result agrees with that obtained when the general solution to the problem of a pulse diffrated by a conducting wedge treated in EM-43[7] is specialized to the case considered here. As expected this result also agrees with that obtained by Keller and Blank, for their case, by a different method.[8]

Then we have

$$u(\eta, \phi) = \frac{2}{\pi}\tan^{-1}\frac{\eta}{\sqrt{-\phi}} \qquad (1.213)$$

1.3 Solution of the Second Characteristic Boundary Value Problem

We start as before with the assumption

$$u(\eta,\phi) = \int_L A(k) e^{-k\phi} H_k(\eta)\, dk.$$

According to (1.19) we must have

$$\left.\frac{\partial u}{\partial n}\right|_{\eta=\sqrt{2p}} = \int_L A(k) e^{-k\phi} \left[\frac{dH_k(\eta)}{d\eta} - k\eta H_k(\eta) \right]_{\eta=\sqrt{2p}} dk = 0. \text{ according to}$$

We need, therefore, a solution of the second equation of (1.22) which satisfies

$$\left[\frac{dH_k(\eta)}{d\eta} - k\eta H_k(\eta) \right]_{\eta=\sqrt{2p}} = 0\ .$$

The solution we desire is

$$H_k(\eta) = e^{k\eta^2}\left(1-\sqrt{2p}\; k\; e^{2pk}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2}\, d\tau\right), \qquad (1.31)$$

and we assume

$$u(\eta,\phi) = \int_L A(k)e^{k(\eta^2-\phi)}\left(1-\sqrt{2p}\; k e^{2pk}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2}\, d\tau\right)dk,$$

which satisfies the boundary condition

$$u(\eta,0) = \int_L A(k)e^{k\eta^2}\left(1-\sqrt{2p}\; k\; e^{2pk}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2}\, d\tau\right)dk.$$

We observe that the constant 1 satisfies the differential equation (1.21) and the boundary condition $\frac{\partial u}{\partial n}\Big|_{\eta=\sqrt{2p}} = 0$.

Thus we can write

$$u(\eta,\phi) = 1+\int_L A(k)e^{k(\eta^2-\phi)}\left(1-\sqrt{2p}\; k\; e^{2pk}\int_{\sqrt{2p}} e^{-k\tau^2}\, d\tau\right)dk \qquad (1.32)$$

and

$$u(\eta,0)-1 = \int_L A(k)e^{k\eta^2}\left(1-\sqrt{2p}\; k e^{2pk}\int_{\sqrt{2p}}^{\eta} e^{-k\tau^2} d\tau\right) dk = \frac{\sqrt{2p}}{\eta} . \qquad (1.33)$$

If we let $z = \eta^2-2p$ and $B(k) = A(k)e^{2pk}$, this last equation is equivalent to

$$\int_L B(k)e^{kz}\, dk - \sqrt{\frac{p}{2}}\int_L kB(k)e^{kz}\int_0^z \frac{e^{-k\tau}}{\sqrt{\tau+2p}}\, d\tau\, dk = \sqrt{\frac{2p}{z+2p}} .$$

If we proceed formally as before to solve for B(k), we obtain

$$\int_L B(k)e^{kz}dk - \sqrt{\frac{p}{2}}\int_0^z \frac{d\tau}{\sqrt{\tau+2p}}\int_L kB(k)\, e^{k(z-\tau)}dk = \sqrt{\frac{2p}{z+2p}}$$

and finally

$$B(k) = \frac{\sqrt{2p}\displaystyle\int_0^{\infty} \frac{e^{-k\tau}}{\sqrt{\tau+2p}}\, d\tau}{1-\sqrt{\frac{p}{2}}\, k\displaystyle\int_L \frac{e^{-k\tau}}{\sqrt{\tau+2p}}\, d\tau} .$$

However, if we were to continue as before to obtain a Volterra integral equation for our solution, using this value of B(k), we would see that our boundary condition (1.33) will not be satisfied. It is easy to see, nevertheless, that a value for B(k) which does work is just one-half that obtained formally.

The incorrect value of the formal result is evidently due to the fact that the interchange of integral signs in our procedure is not legitimate in this case. We proceed therefore with the value

$$B(k) = \frac{\sqrt{\frac{p}{2}} \int_0^\infty \frac{e^{-k}}{\sqrt{\tau + 2p}} \, d\tau}{1 - \sqrt{\frac{p}{2}} \, k \int_0^\infty \frac{e^{-k\tau}}{\sqrt{\tau + 2p}} \, d\tau} , \tag{1.34}$$

and

$$u(\eta,\phi) = 1 + \sqrt{\frac{p}{2}} \int_L \frac{L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}}{1 - k\sqrt{\frac{p}{2}} \, L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\}} \left\{1 - k\sqrt{\frac{p}{2}} \int_0^z \frac{e^{-k\tau}}{\sqrt{\tau+2p}} \, d\tau\right\} e^{k(z-\phi)} dk. \tag{1.35}$$

By writing $\int_0^z \frac{e^{-k\tau}}{\sqrt{\tau+2p}} \, d\tau = \int_0^\infty - \int_z^\infty \frac{e^{-k\tau}}{\sqrt{\tau+2p}} \, d\tau$, we can split the integral on the right side of (1.35) into two parts, one of which can be evaluated immediately: $\sqrt{\frac{p}{2}} \int_L e^{k(z-\phi)} \, L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\} dk + I(-\phi,z) = \sqrt{\frac{p}{2}} \frac{1}{\sqrt{z-\phi+2p}} + I(-\phi,z)$,

where

$$I(y,z) = \frac{p}{2} \int_L \frac{k L_k\left\{\frac{1}{\sqrt{\tau+2p}}\right\} L_k\left\{\frac{1}{\sqrt{\tau+2p+z}}\right\}}{1 - k\sqrt{\frac{p}{2}} \, L_k\left\{\frac{1}{\sqrt{\tau + 2p}}\right\}} e^{ky} \, dk. \tag{1.36}$$

We can find a Volterra integral equation for $I(y)$ just as before. First, it is necessary to integrate by parts in the numerator and denominator of the integrand of (1.36), so that the expression for $I(y)$ becomes

$$I(y) = \sqrt{\frac{p}{2}} \int_L \frac{L_k\left\{\frac{1}{\sqrt{\tau + z+2p}}\right\}}{1 + \sqrt{\frac{p}{2}} \, L_k\left\{\frac{1}{(\tau+2p)^{3/2}}\right\}} e^{ky} \, dk - \frac{p}{2} \int_L \frac{L_k\left\{\frac{1}{(\tau+2p)^{3/2}}\right\} L_k\left\{\frac{1}{\sqrt{\tau+z+2p}}\right\}}{1 + \sqrt{\frac{p}{2}} \, L_k\left\{\frac{1}{(\tau+2p)^{3/2}}\right\}} e^{ky} \, dk.$$

We now take a Laplace transform of both sides of the equation, clear fractions, and take the inverse Laplace transform, applying the faltung theorem as before; we obtain

$$I(y) + \sqrt{\frac{p}{2}} \int_0^y \frac{I(\tau)\,d\tau}{(y-\tau+2p)^{3/2}} = \sqrt{\frac{p}{2}}\ \frac{1}{\sqrt{y+z+2p}} - \frac{p}{2}\int_0^y \frac{d\tau}{(y-\tau+2p)^{3/2}\sqrt{\tau+z+2p}} ,$$

that is,

$$I(y) + \sqrt{\frac{p}{2}} \int_0^y \frac{I(\tau)\,d\tau}{(y+2p-\tau)^{3/2}} = p\left\{\frac{1}{\eta\sqrt{y+2p}} - \frac{\sqrt{y+2p}}{\eta(\eta^2+y+2p)} + \frac{\sqrt{2p}}{\sqrt{y+\eta^2}(\eta^2+y+2p)}\right\}; \quad (1.37)$$

the whole solution, we remember, is given by

$$u(\eta,\phi) = 1 + \sqrt{\frac{p}{2(\eta^2-\phi)}} + I(-\phi) . \qquad (1.38)$$

We observe that according to (1.37),

$$I(o) = \frac{1}{\eta}\sqrt{\frac{p}{2}} ,$$

so that along with (1.38) we have $u(\eta,0) = 1 + \frac{\sqrt{2p}}{\eta}$, as required.

To check the boundary condition $\left.\frac{\partial u}{\partial n}\right|_{\eta=\sqrt{2p}} = 0$ we require the expression $\left.\frac{\partial I}{\partial n}\right|_{\eta=\sqrt{2p}} = \psi$, which is the solution of the integral equation obtained from (1.37) by differentiating and setting $\eta = \sqrt{2p}$, namely

$$\Psi + \sqrt{\frac{p}{2}} \int_0^{-\phi} \frac{\psi\,d\tau}{(2p-\phi-\tau)^{3/2}} = \frac{-p\phi}{(4p-\phi)^2}\sqrt{2p-\phi} - \frac{p}{2(2p-\phi)^{3/2}} ,$$

It can be verified that the only solution of this equation is

$$\psi = \frac{p}{2}(2p-\phi)^{-3/2}$$

If we compute the normal derivative of u from equation (1.38) for $\eta = \sqrt{2p}$ and insert the above value for ψ, we find that $\left.\frac{\partial u}{\partial n}\right|_{\eta=\sqrt{2p}} = 0$, as required.

By a procedure similar to the one employed in Section 1.2 it is a matter of straightforward calculations to show that $u(\eta,\phi)$ given by (1.38) and (1.37) satisfies the differential equation (1.21).

We can solve (1.37) as before by iteration. The result converges rapidly toward the correct solution, provided again that $\frac{-\phi}{4p}$ is not too large. For example, zero-order approximation in this case will be

$$u_o(\eta,\phi) = 1 + \sqrt{\frac{p}{2(\eta^2-\phi)}} + \frac{p}{\eta\sqrt{2p-\phi}} - \frac{p\sqrt{2p-\phi}}{\eta(2p-\phi+\eta^2)} + \frac{p\sqrt{2p}}{\sqrt{\eta^2-\phi}\,(2p-\phi+\eta^2)} . \quad (1.39)$$

2. The Paraboloid of Revolution

2.1 Formulation of the Characteristic Boundary Value Problem

In this case the parabolic cylinder is replaced by a paraboloid of revolution given by

$$x^2 + y^2 = 4p(p-z) , \tag{2.11}$$

where p is again the focal length. The wave equation, similarly in three dimensions, is

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0 . \tag{2.12}$$

In place of the Cartesian coordinates (x,y,z,t) we shall again use a coordinate system adapted to the special shape of our obstacle, the paraboloid of revolution, and to the characteristic surface representing the reflected wave front. We write, therefore,

$$\begin{aligned} \phi &= \sqrt{x^2+y^2+z^2} - (t+p) , & t &= \tfrac{1}{2}(\xi^2+\eta^2) - (\phi+p) \\ \eta^2 &= \sqrt{x^2+y^2+z^2} + z , & x &= \xi\eta\cos\Psi \\ \xi^2 &= \sqrt{x^2+y^2+z^2} - z, & y &= \xi\eta\sin\Psi \\ \Psi &= \tan^{-1}\frac{y}{x} , & z &= \tfrac{1}{2}(\xi^2-\eta^2) . \end{aligned} \tag{2.13}$$

In these coordinates the paraboloid of revolution becomes

$$\eta = \sqrt{2p} . \tag{2.14}$$

Because of symmetry we may assume the wave equation (2.12) to be independent of ψ; then

$$u_{\eta\eta} + u_{\xi\xi} + 2\eta u_{\phi\eta} + 2\xi u_{\phi\xi} + \frac{1}{\eta} u_\eta + \frac{1}{\xi} u_\xi + 4u_\phi = 0. \tag{2.15}$$

We can show, in a manner completely analogous to that employed in the case of the parabolic cylinder, that the reflected wave front, which is at each instant of time a cross section of the four-dimensional characteristic hypercone whose vertex is at the focus of the paraboloid of revolution, is a sphere whose center is at the focus. This time the discontinuity conditions show that the discontinuity across the spherical wave front reflected from the paraboloid varies in such a manner along a ray that it has the form

$$[u] = \frac{f(\theta,\phi)}{r} , \qquad (2.16)$$

where (r,θ,ϕ) are ordinary polar coordinates with the coordinate system center at the focus. The function $f(r,\theta)$ can be determined by inserting in (2.16) the values of $[u]$ and r on the paraboloid. The equation of the parabloid, in polar coordinates, is given by

$$r = \frac{2p}{1 + \cos\theta} \qquad (2.17)$$

and for the boundary condition $u = 0$, $[u] = 1$ on the surface. It follows that $f(r,\theta) = \frac{2p}{1 + \cos\theta}$ and $[u] = \frac{2p}{r(1 + \cos\theta)}$. In paraboloidal coordinates, $[u] = \frac{2p}{\eta^2}$ and it again appears that $[u]$ depends on η alone. We now have

$$u(\eta,\xi,0) = 1 - \frac{2p}{\eta^2} , \qquad (2.18)$$

and we can assume once more that u is independent of ξ. With this assumption (2.15) reduces to

$$u_{\eta\eta} + 2\eta u_{\phi\eta} + \frac{1}{\eta} u_{\eta} + 4u_{\phi} = 0 . \qquad (2.19)$$

The first boundary value problem for this case is then to obtain a solution $u(\eta,\phi)$ of (2.19) in the region defined by $-\infty < \phi \leq 0$, $0 \leq \eta < \infty$ such that $u(\eta,0) = 1 - \frac{2p}{\eta^2}$ and $u(\sqrt{2p},\phi) = 0$.

The second boundary value problem can be set up in a way exactly analogous to the second boundary value problem associated with the parabolic cylinder. The boundary condition on the paraboloid of revolution is

$$\frac{\partial u}{\partial n}\Big|_{\eta=\sqrt{2p}} = \left[\sqrt{2p}\,\frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \eta}\right]_{\eta=\sqrt{2p}} = 0 .$$

By the same argument used before we shall have $1 + \frac{2p}{\eta^2}$ for u on the reflected wave front. We must determine a solution $u(\eta,\phi)$ of (2.19) in the region determined by $-\infty < \phi \leq 0$, $0 \leq \eta < \infty$ such that $\frac{\partial u}{\partial n}\Big|_{\eta=\sqrt{2p}} = 0$ and $u(\eta,0) = 1 + \frac{2p}{\eta^2}$.

2.2 Solution of the First Characteristic Boundary Value Problem

Equation (2.19) separates into

$$\Phi(\emptyset) = -k\,\Phi(\emptyset) \qquad (2.21)$$

and

$$H_k''(\eta) - 2k\eta H_k'(\eta) + \frac{1}{\eta} H_k'(\eta) - 4k\, H_k(\eta) = 0 \, .$$

The solution of the second equation of (2.21) which vanishes at $\eta = \sqrt{2p}$ is

$$H_k(\eta) = e^{k\eta^2} \int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau \, . \qquad (2.22)$$

We therefore assume that

$$u(\eta,\emptyset) = 2\int_L A(k) e^{k(\eta^2-\emptyset)} \int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau\, dk \qquad (2.23)$$

and, in accordance with the boundary condition on the reflected wave front, that

$$u(\eta,0) = 2\int_L A(k) e^{k\eta^2} \int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau\, dk = 1 - \frac{2p}{\eta^2} \, . \qquad (2.24)$$

The integral equation (2.24) can be solved in exactly the same way as the corresponding integral equation associated with the parabolic cylinder. Writing $z = \eta^2 - 2p$, we have

$$\int_L A(k) e^{kz} \int_0^z \frac{e^{-k\tau}}{\tau + 2p}\, d\tau\, dk = 1 - \frac{2p}{z+2p} \, .$$

Continuing formally, we find

$$\int_0^z \frac{d\tau}{\tau + 2p} \int_L A(k) e^{k(z-\tau)}\, d\tau = 1 - \frac{2p}{z+2p}$$

from which, using the faltung theorem, we obtain

$$A(k) = \frac{1 - 2pk\int_0^{\infty} \frac{e^{-k\tau}}{\tau + 2p}\, d\tau}{k\int_0^{\infty} \frac{e^{-k\tau}}{\tau + 2p}\, d\tau} \, . \qquad (2.25)$$

On substituting this result into (2.23) we obtain

$$u(\eta,\emptyset) = \int_L \left\{ \frac{1}{kL_k\left\{\frac{1}{\tau+2p}\right\}} - 2p \right\} e^{k(\eta^2-2p)} \int_{2p}^{\eta^2} \frac{e^{-k\tau}}{\tau}\, d\tau\, dk$$

$$= \int_L \left\{ \frac{1}{kL_k\left\{\frac{1}{\tau+2p}\right\}} - 2p \right\} e^{k(\eta^2-\emptyset)} \left\{ \int_{2p}^{\infty} - \int_{2}^{\infty} \frac{e^{-k\tau}}{\tau}\, d\tau \right\} dk$$

$$= \int_L \frac{e^{k(\eta^2-\emptyset-2p)}}{k}\, dk \;-2p \int_L L_k\left\{\frac{1}{\tau+2p}\right\} e^{k(\eta^2-\emptyset-2p)} dk$$

$$+ 2p \int_L L_k\left\{\frac{1}{\tau+\eta^2}\right\} e^{-k\emptyset}\, dk - I(-\emptyset) \;,$$

or, finally,

$$u(\eta,\emptyset) = 1 - I(-\emptyset), \tag{2.26}$$

where
$$I(y) = \int_L \frac{e^{ky} L_k\left\{\frac{1}{\tau+\eta^2}\right\}}{kL_k\left\{\frac{1}{\tau+2p}\right\}}\, dk \,. \tag{2.27}$$

We can now obtain a Volterra integral equation for I(y) by the procedure we have used before. Taking the Laplace transform of both sides of (2.27) and clearing fractions, we obtain

$$L_k\left\{\frac{1}{\tau+2p}\right\} L_k\left\{I(\tau)\right\} = L_k\left\{1\right\} L_k\left\{\frac{1}{\tau+\eta^2}\right\} .$$

By the faltung theorem, then,

$$\int_0^y \frac{I(\tau)}{(y+2p-\tau)}\, d\tau = \int_0^y \frac{d\tau}{\tau+\eta^2} \;,$$

and differentiating with respect to y we have

$$I(y) - 2p \int_0^y \frac{I(\tau)\, d\tau}{(y+2p-\tau)^2} = \frac{2p}{y+\eta^2} \,. \tag{2.28}$$

From (4.28) we have $I(o) = \frac{2p}{\eta^2}$ which, combined with (2.26), shows $u(\eta,0) = 1 - \frac{2p}{\eta^2}$, as desired.

As before we can show that $u(\eta,\phi)$ as given by (2.26) with $I(y)$ defined by (2.28) satisfies our differential equation and the boundary condition $u(\sqrt{2p},\phi) = 0$.

We may obtain I by iterating (2.28) as before. The solution will again be valid in the neighborhood of the reflected wave front, with the region of validity varying inversely as the focal length p.

2.3 Solution of the Second Characteristic Boundary Value Problem

We now consider the boundary condition $\frac{\partial u}{\partial n} = 0$ on the paraboloid of revolution. Once again we have $\left.\frac{\partial u}{\partial n}\right|_{\eta=\sqrt{2p}} = \left[\sqrt{2p}\,\frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \eta}\right]_{\eta=\sqrt{2p}} = 0$.

In place of $u(\eta,0) = 1-\frac{2p}{\eta^2}$ we have $u(\eta,0) = 1 + \frac{2p}{\eta^2}$ on the wave front.

The solution of the second equation of (2.21) required in this case is

$$H_k(\eta) = \left[1-2pk\, e^{2pk}\int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau\right] e^{k\eta^2} .$$

Thus,

$$u(\eta,\phi) = 1 + \int_L A(k)e^{k(\eta^2-\phi)}\left[1-2pke^{2pk}\int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau\right] dk \qquad (2.31)$$

and

$$u(\eta,0) = 1 + \int_L A(k)\, e^{k\eta^2}\left[1-2pke^{2pk}\int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau\right] dk = \frac{2p}{\eta^2} + 1 ,$$

that is,

$$\int_L A(k)\, e^{k\eta^2}\left[1-2pke^{2pk}\int_{\sqrt{2p}}^{\eta} \frac{e^{-k\tau^2}}{\tau}\, d\tau\right] dk = \frac{2p}{\eta^2} . \qquad (2.32)$$

We must solve for A(k) in (2.32). We proceed formally as in the previous cases, setting $\eta^2-2p = z$ and $B(k) = A(k)e^{2pk}$ and changing variables in the integral

$$\int_L B(k)e^{kz}\left[1-pk\int_0^z \frac{e^{-k\tau}}{\tau+2p}\, d\tau\right] dk = \frac{2p}{z+2p} ,$$

so that

$$\int_L B(k)e^{kz}\,dk - p\int_0^z \frac{d\tau}{\tau+2p}\int_L kB(k)e^{k(z-\tau)}\,dk = \frac{2p}{z+2p},$$

whereupon we have the formal result

$$B(k) = \frac{2p\int_0^\infty \frac{e^{-k\tau}}{\tau+2p}\,d\tau}{1-2p\int_0^\infty \frac{e^{-k\tau}}{\tau+2p}\,d\tau}.$$

But in this case, as in a previous one, this value of B(k) is incorrect, and for the same reason. However, one-half the above result works, as we can demonstrate: We set

$$u(\eta,\phi) = 1 + p\int_L \frac{L_k\left\{\frac{1}{\tau+2p}\right\}}{1-pk\,L_k\left\{\frac{1}{\tau+2p}\right\}}\, e^{k(z-\phi)}\left\{1-pk\int_0^z \frac{e^{-k\tau}}{\tau+2p}\,d\tau\right\}dk$$

$$= 1 + \frac{p}{z+2p-\phi} + pI(-\phi),$$

that is,

$$u(\eta,\phi) = 1 + \frac{p}{\eta^2-\phi} + pI(-\phi), \tag{2.33}$$

where

$$I(y) = p\int_L \frac{kL_k\left\{\frac{1}{\tau+2p}\right\}e^{ky}}{1-pk\,L_k\left\{\frac{1}{\tau+2p}\right\}}\,L_k\left\{\frac{1}{\tau+\eta^2}\right\}dk$$

$$= p\int_L \frac{\left[\frac{1}{p} - 2L_k\left\{\frac{1}{(\tau+2p)^2}\right\}\right]e^{ky}L_k\frac{1}{\tau+\eta^2}}{1+2p\,L_k\left\{\frac{1}{(\tau+2p)^2}\right\}}\,dk. \tag{2.34}$$

Then, taking Laplace transforms and applying the faltung theorem, we obtain

$$I + 2p\int_0^y \frac{I(\tau)\,d\tau}{(y-\tau+2p)^2} = \frac{1}{\eta^2+y} - 2p\int_0^y \frac{d\tau}{(\tau+2p)^2\,(y+\eta^2-\tau)},$$

that is,

$$I(y) + 2p\int_0^y \frac{I(\tau)\,d\tau}{(y+2p-\tau)^2} = \frac{1}{y+\eta^2} - \frac{y}{(2p+y)(\eta^2+y+2p)} - \frac{2p\log\left[\frac{(2p+y)(\eta^2+y)}{2p\,\eta^2}\right]}{(\eta^2+y+2p)^2}. \tag{2.35}$$

It follows that $I(o) = \frac{1}{\eta^2}$, so that by (4.33), $u(\eta,0) = 1 + \frac{2p}{\eta^2}$ as required.

We can show that

$$\frac{\partial u}{\partial n}\bigg|_{\eta=\sqrt{2p}} = \left[\sqrt{2p}\,\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial n}\right]_{\eta=\sqrt{2p}} = 0 \text{ and that u}$$

and that u satisfies (2.19) when u is defined by (2.33) and (2.35) in the same way as in the case of the parabolic cylinder. The equation (2.35) can be solved by iteration as before.

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